

# Reduction of Coupling Parameters in Quantum Field Theories <sup>1</sup>

Reinhard Oehme <sup>2</sup>

*Enrico Fermi Institute and Department of Physics  
University of Chicago  
Chicago, Illinois, 60637, USA*

## Abstract

A concise survey is given of the general method of reduction in the number of coupling parameters. Theories with several independent couplings are related to a set of theories with a single coupling. The reduced theories may or may not have particular symmetries. A few have asymptotic power series expansions, others contain non-integer powers and/or logarithmic factors. An example is given with two power series solutions, one with  $N = 2$  Supersymmetry, and one with no known symmetry. In a second example, the reduced Yukawa coupling of the superpotential in a dual magnetic supersymmetric gauge theory is uniquely given by the square of the magnetic gauge coupling with a known factor.

---

<sup>1</sup>For the ‘*Concise Encyclopedia of SUPERSYMMETRY*’,  
Kluwer Academic Publishers, Dordrecht, (Editors: Jon Bagger, Steven Duplij and Warren Siegel)  
2001.

<sup>2</sup>E-mail: oehme@theory.uchicago.edu

REDUCTION OF COUPLING PARAMETERS in multi parameter quantum field theories [1,2] often leads to one-parameter theories with supersymmetry or other symmetries. In addition, there may be solutions with no recognizable symmetry. For pairs of dual supersymmetric gauge theories, which require the presence of superpotentials, the reduction method makes it possible to obtain dual pairs with each theory having a single coupling parameter[3]. There are many other applications of the reduction Method [4,5].

Consider quantum field theories with several dimensionless coupling parameters

$$\lambda, \lambda_1, \dots, \lambda_n.$$

The corresponding effective couplings  $\bar{\lambda}(u), \bar{\lambda}_k(u), k = 1, \dots, n$  are functions of scaling parameter  $u = k^2/\kappa^2$ , where  $\kappa^2 < 0$  is the normalization point. They satisfy the renormalization group equations

$$u(d\bar{\lambda}/du) = \beta(\bar{\lambda}, \bar{\lambda}_1, \dots, \bar{\lambda}_n), \quad u(d\bar{\lambda}_k/du) = \beta_k(\bar{\lambda}, \bar{\lambda}_1, \dots, \bar{\lambda}_n),$$

where the functions  $\beta$  and  $\beta_k$  are obtained from the vertex terms of the theory. With  $\bar{\lambda}(u)$  being an analytic function, one can choose a point where  $(d\bar{\lambda}(u)/du) \neq 0$  and introduce  $\bar{\lambda}(u)$  as a new variable in these equations. With  $\bar{\lambda}_k(u) \rightarrow \lambda_k(\bar{\lambda}(u))$ , and  $\bar{\lambda}(u) \rightarrow \lambda$ , this substitution eliminates the variable  $u$  and yields the REDUCTION EQUATIONS

$$\beta(\lambda) \frac{d\lambda_k(\lambda)}{d\lambda} = \beta_k(\lambda), \quad k = 1, \dots, n. \quad (1)$$

Here  $\beta(\lambda) = \beta(\lambda, \lambda_1, \dots, \lambda_n)$  and  $\beta_k(\lambda) = \beta_k(\lambda, \lambda_1, \dots, \lambda_n)$ , with the insertions  $\lambda_k = \lambda_k(\lambda)$ ,  $k = 1, \dots, n$ . The reduction equations are necessary and sufficient for the Green's functions of the one-parameter theory

$$G(k_i, \kappa^2, \lambda) = G(k_i, \kappa^2, \lambda, \lambda_1(\lambda), \dots, \lambda_n(\lambda))$$

to satisfy the renormalization group equations in the single variable  $\lambda$ :

$$\left( \kappa^2 \frac{\partial}{\partial \kappa^2} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_G(\lambda) \right) G(k_i, \kappa^2, \lambda) = 0, \quad (2)$$

where  $\beta(\lambda)$  and  $\gamma_G(\lambda)$  are again given by the corresponding coefficients of the multi-parameter theory with insertions. Of course, one can also require the validity of eq.(2) in order to obtain the reduction equations (1). A priori, the reduction scheme is very general. But for most applications considered, the functions  $\lambda_k(\lambda)/\lambda = f_k(\lambda)$  are bounded for  $\lambda \rightarrow 0$ . Furthermore, the  $\beta$ -functions are represented by asymptotic power series in the weak coupling limit:

$$\beta(\lambda, \lambda_1, \dots, \lambda_n) = \beta_0 \lambda^2 + (\beta_1 \lambda^3 + \beta_{1k} \lambda_k \lambda^2 + \beta_{1kk'} \lambda_k \lambda_{k'} \lambda) + \dots, \quad (3)$$

$$\beta_k(\lambda, \lambda_1, \dots, \lambda_n) = (c_k^{(0)}\lambda^2 + c_{k,k'}^{(0)}\lambda_{k'}\lambda + c_{k,k'k''}^{(0)}\lambda_{k'}\lambda_{k''}) + \dots \quad (4)$$

It is seen that the the reduction equations are singular at the origin. This implies that the Picard-Lindelof theorem about the uniqueness of solutions does not apply. Using equivalence transformations, possible mass and gauge parameter dependencies of the coefficient functions can be removed. With the original  $\beta$ -functions given as asymptotic power series expansions, solutions  $\lambda_k(\lambda)$  of the reduction equations are considered which are also of the form of asymptotic expansions. Of special interest are solutions in the form of power series expansions, but in general, non-integer powers as well as logarithmic terms are possible. Consider first power series solutions

$$\lambda_k(\lambda) = \lambda f_k(\lambda), \quad f_k(\lambda) = f_k^0 + \sum_{m=1}^{\infty} \chi_k^{(m)} \lambda^m. \quad (5)$$

Substitution into the reduction equations yields the fundamental one-loop relation

$$c_k^{(0)} + (c_{kk'}^{(0)} - \beta_0 \delta_{kk'}) f_{k'}^0 + c_{kk'k''}^{(0)} f_{k'}^0 f_{k''}^0 = 0. \quad (6)$$

Given a solution  $f_k^0$  of these quadratic equations, the one-loop criteria

$$\det(M_{kk'}(f^0) - m\beta_0 \delta_{kk'}) \neq 0 \quad \text{for } m = 1, 2, \dots, \quad (7)$$

$$M_{kk'}(f^0) = c_{k,k'}^{(0)} + 2c_{k,k'k''}^{(0)} f_{k''}^0 - \delta_{kk'} \beta_0. \quad (8)$$

are sufficient to insure that all coefficients  $\chi^{(m)}$  in the expansion of  $f_k(\lambda)$  are determined. Then the reduced theory has a power series expansion in  $\lambda$ , and all possible solutions of this kind are determined by the one-loop equation for  $f_k^0$ . With the coefficients  $\chi^{(m)}$  fixed, one can use regular reparametrization transformations in order to remove all but the first term in the expansion of the functions  $f_k(\lambda)$ . These reparametrization transformations are of the form

$$\begin{aligned} \lambda' &= \lambda'(\lambda, \lambda_1, \dots, \lambda_n) = \lambda + a^{(20)}\lambda^2 + a_k^{(11)}\lambda_k\lambda + \dots, \\ \lambda'_k &= \lambda'_k(\lambda, \lambda_1, \dots, \lambda_n) = \lambda_k + b_{kk'k''}^{(20)}\lambda_{k'}\lambda_{k''} + b_{kk'}^{(11)}\lambda_{k'}\lambda + \dots. \end{aligned}$$

They leave one-loop quantities invariant. Given the validity of the conditions (7), there is then a frame where the solutions are of the form

$$\lambda_k(\lambda) = \lambda f_k^0, \quad (9)$$

with the coefficients  $f_k^0$  determined by the one-loop reduction equations (6). These usually have only a few characteristic solutions. In the special case where  $f_k^0 = 0$ , and  $\chi_k^{(m)} = 0$  for  $m < N$ , one has  $f_k(\lambda) = \chi_k^{(N)}\lambda^N$  after an appropriate reparametrization.

Besides the power series solutions, reduced to the form (9), there can be ‘general’ solutions of eqs.(1) which approach the same limit  $f_k^0$ , but contain non-integer powers. For example, if the matrix  $\beta_0^{-1}M(f^0)$  has one non-integer eigenvalue  $\eta > 0$ , then there is a solution  $f_k(\lambda) = f_k^0 + \chi_k^{(\eta)}\lambda^\eta + \dots$ , after reparametrization. The coefficient  $\chi_k^{(\eta)}$  contains  $r$  free parameters if the eigenvalue has  $r$ -fold degeneracy. All other coefficients are determined. Questions about the stability of solutions have been discussed in connection with the Lyapunov-Malkin theorems [3].

In case the determinant (7) should vanish due to some positive eigenvalue  $m = N$ , then the asymptotic series solution contains in general terms like  $\lambda^N \log \lambda$ .

The essential features of the reduction method are best seen by evaluating cases of particular interest. For a reduction resulting in SUSY and non-SUSY theories, one can consider a gauge theory with one Dirac field, one scalar and one pseudoscalar field, all in the adjoint representation of SU(2) [1,2]. Besides the usual gauge couplings, the direct interaction part of the Lagrangian is given by

$$\begin{aligned} L_{dir.int.} &= i\sqrt{\lambda_1} \epsilon^{abc} \bar{\psi}^a (A^b + i\gamma_5 B^b) \psi^c \\ &- \frac{1}{4} \lambda_2 (A^a A^a + B^a B^a)^2 + \frac{1}{4} \lambda_3 (A^a A^b + B^a B^b)^2 . \end{aligned} \quad (10)$$

Writing  $\lambda = g^2$ , where  $g$  is the gauge coupling, and  $\lambda_k = \lambda f_k$ , with  $k=1,2,3$ , the one-loop  $\beta$ -function coefficients of this theory are

$$(16\pi^2)\beta_{g^0} = -4, \quad (16\pi^2)\beta_1^0 = 8f_1^2 - 12f_1, \quad (16\pi^2)\beta_2^0 = 3f_3^2 - 12f_3f_2 + 14f_2^2 + 8f_1f_2 - 8f_1^2 - 12f_2 + 3, \quad (16\pi^2)\beta_3^0 = -9f_3^2 + 12f_3f_2 + 8f_3f_1 - 12f_3 - 3.$$

The algebraic reduction equations (4) have four real solutions:

$$(f_1^0 = 1, f_2^0 = 1, f_3^0 = 1), \quad (f_1^0 = 1, f_2^0 = \frac{9}{\sqrt{105}}, f_3^0 = \frac{7}{\sqrt{105}}),$$

and two others with reversed signs of  $f_2^0$  and  $f_3^0$ , so that the classical potential approaches negative infinity with increasing magnitude of the scalar fields. These latter solutions will not be considered further.

The eigenvalues of the matrix  $\beta_{g^0}^{-1}M(f^0)$  are respectively  $(-2, -3, +\frac{1}{2})$  and  $(-2, -\frac{3}{4} \frac{25+\sqrt{343}}{\sqrt{105}}, -\frac{3}{4} \frac{25-\sqrt{343}}{\sqrt{105}}) = (-2, -3.189..., -0.470...)$ .

There are no positive integers appearing in these expressions. Hence the coefficients of the power series solutions are determined and can be removed by reparametrization, except for the invariant first term. Then the solutions are

$$(a) \quad \lambda_1 = \lambda_2 = \lambda_3 = g^2, \quad (11)$$

which corresponds to an  $N = 2$  extended SUSY Yang-Mills theory, and

$$(b) \quad \lambda_1 = g^2, \quad \lambda_2 = \frac{9}{\sqrt{105}} g^2, \quad \lambda_3 = \frac{7}{\sqrt{105}} g^2, \quad (12)$$

which is not associated with any known symmetry, at least in four dimensions. Both theories are ‘minimally’ coupled gauge theories with matter fields. The eigenvalues of the matrix  $\beta_{g_0}^{-1} M(f^0)$  are all negative with the exception of the third one for the  $N=2$  supersymmetric theory. In this case there exists a general solution with  $\eta = +\frac{1}{2}$ , and with the coefficient given by  $\chi^{(\frac{1}{2})} = (0, C, 3C)$ , where  $C$  is an arbitrary parameter. The theory with  $C \neq 0$  corresponds to one with hard breaking of SUSY. It has an asymptotic power series in  $g$  and not in  $g^2$ , as is the case for the invariant theory.

From the present example, and many others, one realizes that the special frame, where the power series solutions of the reduction equations are of the simple form (9), is a natural frame as far as the reduced one-parameter theories are concerned. The  $\beta$ -functions of the reduced theories are still power series and are not reduced to polynomials.

As another application of the reduction method, the “magnetic” gauge theory is considered which is the dual of SQCD as the “electric” theory [3]. For SQCD the gauge group is  $SU(N_C)$  with  $N = 1$  SUSY, and there are  $N_F$  quark superfields in the fundamental representation. There is only one coupling parameter, the gauge coupling  $g_e$ . The  $\beta$ -function is given by the asymptotic expansion

$$\beta_e(g_e^2) = \beta_{e0} g_e^4 + \beta_{e1} g_e^6 + \cdots,$$

with one loop coefficient

$$\beta_{e0} = (16\pi^2)^{-1}(-3N_C + N_F).$$

It is proposed that there exists a dual magnetic theory which provides an alternate description at low energies [6]. But both theories coincide only at the non-trivial infrared fixed point in the conformal window  $\frac{2}{3}N_C < N_F < 3N_C$ . For appropriate values of  $N_C$  and  $N_F$ , the magnetic theory has the gauge group  $G^d = SU(N_C^d)$  with  $N_C^d = N_F - N_C$ . There are  $N_F^d = N_F$  quark superfields  $q$  in the fundamental representation, the corresponding anti-quark superfields  $\bar{q}$ , and  $N_F^2$  independent scalar superfields  $M$ , which are coupled via a Yukawa superpotential of the form  $\sqrt{\lambda} M_j^i q_i \bar{q}^j$ . This coupling is required by the anomaly matching conditions, which are used in the construction of dual theories, and by the need for both theories to have the same physical symmetries. In the conformal window, the potential drives

the magnetic theory to the infrared fixed point. The  $\beta$ -functions of the magnetic theory can be written in the form

$$\begin{aligned}\beta_m(g_m^2, \lambda) &= \beta_{m0} g_m^4 + (\beta_{m1} g_m^6 + \beta_{m1,\lambda} g_m^4 \lambda) + \dots \\ \beta_\lambda(g_m^2, \lambda) &= c_\lambda g_m^2 \lambda + c_{\lambda\lambda} \lambda^2 + \dots,\end{aligned}$$

with the relevant lowest order coefficients given by

$$\begin{aligned}\beta_{m0} &= (16\pi^2)^{-1}(3N_C - 2N_F) \\ c_\lambda &= (16\pi^2)^{-1} \left( -4 \frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)} \right) \\ c_{\lambda\lambda} &= (16\pi^2)^{-1} (3N_F - N_C).\end{aligned}$$

At first, the reduction method is applied to the magnetic theory in the conformal window  $\frac{2}{3}N_C < N_F < 3N_C$ . There are two power series solutions. After reparametrization, one solution is given by [3]

$$\lambda_1(g_m^2) = g_m^2 f(N_C, N_F), \quad (13)$$

$$f(N_C, N_F) = \frac{\beta_{m0} - c_\lambda}{c_{\lambda\lambda}} = \frac{N_C (N_F - N_C - 2/N_C)}{(N_F - N_C)(3N_F - N_C)}. \quad (14)$$

The other solution is  $\lambda_2(g_m^2) \equiv 0$ . Since the latter removes the superpotential, it is excluded, and one is left with a unique single power series solution. This solution implies a theory with a single gauge coupling  $g_m$ , and renormalized perturbation expansions which are power series in  $g_m^2$ . It is the appropriate dual of SQCD. There are ‘general’ solutions, but they all approach the excluded power solution  $\lambda_2(g_m^2) \equiv 0$ . With one exception, they involve non-integer powers of  $g_m^2$ . The reduction can be extended to the ‘free electric region’  $N_F > 3N_C$ , and to the ‘free magnetic region’  $N_C + 2 < N_F < \frac{2}{3}N_C$ , which is non-empty for  $N_C > 4$ . The results are similar. In the free magnetic case, one deals however with the approach to a trivial infrared fixed-point.

Further applications of the reduction method in connection with duality may be found in [4]. Dual theories can be obtained as appropriate limits of brane systems. In these brane constructions, duality corresponds essentially to a reparametrization of the quantum moduli space of vacua of a given brane structure. It remains to find out how the reduction solutions are related to special features of these constructions. There are also more phenomenological uses for the reduction schemes, in particular within the framework of supersymmetric grand unified theories [5].

## BIBLIOGRAPHY

- [1] R. Oehme and W. Zimmermann, MPI Report MPT-PAE/Pth 60/82 (1982), Commun. Math. Phys. **97** (1985) 569; R. Oehme, K. Sibold and W. Zimmermann, Phys. Lett. **B147** (1984) 115; Phys. Lett. **B153** (1985) 142.
- [2] W. Zimmermann, Commun. Math. Phys. **97** (1985) 211 (Symanzik Memorial Volume); R. Oehme, CERN-Report TH.42-45/1985, Prog. Theor. Phys. Suppl. **86** (1986) 215 (Nambu Festschrift); K. Sibold, Acta Physica Polonica, **19** (1988) 295. (These papers contain reference to earlier work on reduction methods)
- [3] R. Oehme, Phys. Lett. **B399** (1997) 67; Phys. Rev. **D59** (1999) 105004; R. Oehme, ‘Reduction of Coupling Parameters and Duality’ in *Recent Developments in Quantum Field Theory*, edited by P. Breitenlohner, D. Maison and J. Wess (Springer Verlag, Heidelberg, New York, 2000), hep-th/9903092.
- [4] E. Gardi and G. Grunberg, JHEP **024** (1999) 9903.
- [5] T. Kobayashi, J. Kubo, M. Mondragon and G. Zoupanos, Nucl. Phys. **B511** (1998) 45; J. Kubo, ‘Applications of Reduction of Couplings’, in *Recent Developments in Quantum Field Theory*, edited by P. Breitenlohner, D. Maison and J. Wess (Springer Verlag, Heidelberg, New York, 2000), hep-th/9903482, (This article contains detailed references to phenomenological applications).
- [6] N. Seiberg, Phys. Rev. **D49** (1994) 6857; Nucl. Phys. **B435** (1996) 129.

*Reinhard Oehme*